

Dressing the giant magnon II

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ABSTRACT: We extend earlier work by demonstrating how to construct classical string solutions describing arbitrary superpositions of scattering and bound states of dyonic giant magnons on S^5 using the dressing method for the $SU(4)/Sp(2)$ coset model. We present a particular scattering solution which generalizes solutions found in [hep-th/0607009](#) and [hep-th/0607044](#) to the case of arbitrary magnon momenta. We compute the classical time delay for the scattering of two dyonic magnons carrying angular momenta with arbitrary relative orientation on the S^5 .

KEYWORDS: AdS-CFT Correspondence, Gauge-gravity correspondence, Integrable Field Theories.

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1. Introduction

The study of classical spinning string solutions in $AdS_5 \times S^5$ has provided a wealth of data for detailed study of the AdS/CFT correspondence. An interesting step forward was recently taken by Hofman and Maldacena [1], who found the classical string solution corresponding to a single magnon in the dual gauge theory. In this context the word magnon refers to an elementary excitation which can travel along a chain of Z 's with some momentum p , i.e.

$$\mathcal{O}_p \sim \sum_l e^{ipl} (\dots ZZZWZZZ\dots), \tag{1.1}$$

where the magnon W is inserted at position l along the chain. The corresponding ‘giant magnon’ is an open string whose endpoints move at the speed of light along an equator of the S^5 , separated in longitude by an angle p . This state carries an infinite amount of angular momentum J in the plane of the equator of the S^5 and is characterized by a finite value of $\Delta - J$. Recent papers on giant magnons include [2–18].

In [8] we employed the dressing method [19–21] to construct classical string solutions corresponding to various scattering and bound states of magnons. In particular, we demonstrated how to obtain solutions representing superpositions of any number of elementary giant magnons (or bound states thereof) on $\mathbb{R} \times S^5$, as well as any number of dyonic giant magnons on $\mathbb{R} \times S^3$. The dyonic giant magnon, discovered in [2, 3], is a BPS bound state of many ($\mathcal{O}(\sqrt{\lambda})$) magnons which carries, in addition to an infinite amount of J in the equator of the S^3 , a non-zero macroscopic amount of angular momentum J_1 in the orthogonal plane on S^3 .

In this note we study scattering states of dyonic giant magnons on S^5 , some special cases of which have appeared in [8, 9, 17]. We fill a gap in our previous work by demonstrating how to construct classical string solutions describing general scattering states of dyonic giant magnons whose individual angular momenta J_i have arbitrary orientations in the directions transverse to the equator of the S^5 .

After reviewing the basics of giant magnons in section 2, we explain in section 3 how to apply the dressing method for the $SU(4)/Sp(2) = S^5$ coset model, following the construction of [21]. This coset construction apparently has more flexibility than the $SO(6)/SO(5) = S^5$ coset construction employed in [8], since we have been unable to find the dyonic giant magnon solution via the latter dressing method. In [8] the $SU(2)$ principal chiral model was instead used to construct superpositions of dyonic magnons. That was sufficient for solutions living only on an $S^3 \subset S^5$, but the $SU(4)/Sp(2)$ coset used in this paper allows us to construct solutions living on the full S^5 . In section 4 we begin with a detailed analysis of the parameter space for a single soliton in the $SU(4)/Sp(2)$ coset model. We present in (4.13) a particular explicit solution for the scattering of two dyonic giant magnons with arbitrary momenta p_1, p_2 which carry angular momentum in orthogonal planes. This solution generalizes the special case $p_1 = -p_2 = \pi$ which was obtained in [8] and was generalized to $p_1 = -p_2 = p$ in [9]. Finally in section 5 we calculate the classical time delay for the scattering of two dyonic giant magnons with arbitrary relative orientations on the S^5 . It would be interesting to compare the corresponding classical phase shift to a gauge theory analysis along the lines of [1, 13, 14].

2. Giant magnons

We consider string theory on $\mathbf{R} \times S^5$ in conformal gauge, writing the S^5 part of the theory in terms of three complex fields Z_i subject to the constraint

$$Z_i \bar{Z}_i = 1. \tag{2.1}$$

After choosing $X^0 = t$ (X^0 is the time coordinate on $\mathbf{R} \times S^5$), the equation of motion for the Z_i can be written as

$$\bar{\partial} \partial Z_i + \frac{1}{2} (\partial Z_j \bar{\partial} \bar{Z}_j + \partial \bar{Z}_j \bar{\partial} Z_j) Z_i = 0, \tag{2.2}$$

where we use the worldsheet coordinates $z = \frac{1}{2}(x-t)$, $\bar{z} = \frac{1}{2}(x+t)$. The Virasoro constraints take the form

$$\partial Z_i \partial \bar{Z}_i = \bar{\partial} Z_i \bar{\partial} \bar{Z}_i = 1. \tag{2.3}$$

We consider a giant magnon to be any open string whose endpoints move at the speed of light along an equator of the S^5 , which we choose lie in the Z_1 plane. The appropriate boundary conditions at fixed t are

$$\begin{aligned} Z_1(t, x \rightarrow \pm\infty) &= e^{i(t \pm p/2) + i\alpha}, \\ Z_i(t, x \rightarrow \pm\infty) &= 0, \quad i = 2, 3, \end{aligned} \tag{2.4}$$

where $e^{i\alpha}$ is an arbitrary overall phase and p represents the difference in longitude between the endpoints of the string on the equator of the S^5 . In the gauge theory picture, p is identified with the momentum of the magnon [1]. We may refer to p as the ‘momentum’ of a magnon, but it should be kept in mind that the worldsheet momentum density of all of the solutions we consider is zero due to the Virasoro constraints (3.2).

The equations (2.1)–(2.4) have infinitely many distinct solutions, which can be partly classified by their conserved charges. The boundary conditions (2.4) explicitly break the $SO(6)$ symmetry of the S^5 down to $U(1) \times SO(4)$. The conserved charge associated with the $U(1)$ is

$$\Delta - J = \frac{\sqrt{\lambda}}{2\pi} \int_{-\infty}^{+\infty} dx (1 - \text{Im}[\bar{Z}_1 \partial_t Z_1]), \quad (2.5)$$

where $\sqrt{\lambda}/2\pi$ is the string tension expressed in terms of the 't Hooft coupling λ of the dual gauge theory. The $SO(4)$ symmetry leads to conserved angular momentum matrix

$$J_{ab} = i \frac{\sqrt{\lambda}}{2\pi} \int_{-\infty}^{\infty} dx (X_a \partial_t X_b - X_b \partial_t X_a), \quad a, b = 1, \dots, 4, \quad (2.6)$$

which we have written in terms of the real basis defined by $Z_2 = X_1 + iX_2$, $Z_3 = X_3 + iX_4$.

3. Dressing method for $S^5 = SU(4)/Sp(2)$

In order to construct solutions of (2.1)–(2.4) we will apply the dressing method of Zakharov and Mikhailov [19, 20] for building soliton solutions of classically integrable equations, following the application of this method to the $SU(4)/Sp(2)$ coset model given in [21].

We will see that an elementary soliton of the $SU(4)/Sp(2)$ coset model is characterized by the choice of a complex parameter λ and a point w on \mathbb{P}^3 . The most general solution obtainable¹ via the dressing method takes the form of a scattering state of any number of elementary solitons or bound states of them. Each individual soliton carries some ‘momentum’ p_i and a single non-zero $SO(4)$ angular momentum J_i (i.e., the eigenvalues of the matrix (2.6) for a single soliton are $\{+J_i, -J_i, 0, 0\}$). These two quantities are encoded in the parameter λ_i of the soliton, while the parameter w_i determines the plane of its angular momentum in the transverse \mathbf{R}^4 (i.e., the eigenvectors of (2.6)).

The simplest context in which the dressing method may be applied is the reduced [22] principal chiral model describing a unitary $N \times N$ matrix $g(z, \bar{z})$ satisfying the equation of motion

$$\bar{\partial}(\partial g g^{-1}) + \partial(\bar{\partial} g g^{-1}) = 0 \quad (3.1)$$

subject to the Virasoro constraints

$$\frac{1}{N} \text{Tr}[(ig^{-1} \partial g)^2] = 1, \quad \frac{1}{N} \text{Tr}[(ig^{-1} \bar{\partial} g)^2] = 1. \quad (3.2)$$

¹It is not clear to us that all solutions may be obtained through the dressing method. For example, we have been unable to obtain the dyonic giant magnon solution via the dressing method in the $SO(6)/SO(5)$ coset model.

Given any solution $g(z, \bar{z})$ of these equations, the dressing method provides for the construction of an appropriate dressing matrix χ such that

$$g'(z, \bar{z}) = \chi(z, \bar{z})g(z, \bar{z}) \quad (3.3)$$

is also solution of (3.1) and (3.2).

For the application to classical string theory on $\mathbf{R} \times S^5$ we are not interested in a principal chiral model but rather a coset model. In previous work [8] we employed the $S^5 = \text{SO}(6)/\text{SO}(5)$ but for the present analysis it is more fruitful to use the coset $S^5 = \text{SU}(4)/\text{Sp}(2)$ following the analysis of [21]. We define this coset by imposing on $g \in \text{SU}(4)$ the constraint

$$g^T = \mathcal{J}g\mathcal{J}^{-1}, \quad (3.4)$$

where \mathcal{J} is the fixed antisymmetric matrix

$$\mathcal{J} = \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \quad (3.5)$$

A convenient parametrization of this coset, which allows us to immediately read off the S^5 coordinates Z_i from the matrix g , is given by

$$g = \begin{pmatrix} Z_1 & Z_2 & 0 & Z_3 \\ -\bar{Z}_2 & \bar{Z}_1 & -Z_3 & 0 \\ 0 & \bar{Z}_3 & Z_1 & -\bar{Z}_2 \\ -\bar{Z}_3 & 0 & Z_2 & \bar{Z}_1 \end{pmatrix}, \quad (3.6)$$

which is unitary and satisfies (3.4) precisely when (2.1) holds.

To apply the dressing method, we begin with a given solution g by solving the linear system

$$\partial\Psi = \frac{\partial g g^{-1}\Psi}{1-\lambda}, \quad \bar{\partial}\Psi = \frac{\bar{\partial}g g^{-1}\Psi}{1+\lambda} \quad (3.7)$$

to find $\Psi(\lambda)$ as a function of the auxiliary complex parameter λ , subject to the initial condition

$$\Psi(0) = g, \quad (3.8)$$

the unitarity constraint

$$[\Psi(\bar{\lambda})]^\dagger \Psi(\lambda) = 1, \quad (3.9)$$

and the coset constraint

$$\Psi(\lambda) = \Psi(0)\mathcal{J}\overline{\Psi(1/\bar{\lambda})}\mathcal{J}^{-1}, \quad (3.10)$$

whose role is to ensure that the dressed solution g' we now construct will continue to satisfy the coset condition (3.4).

Once we know $\Psi(\lambda)$, the dressing factor for a single soliton may be written in terms of the parameters (λ_i, w_i) discussed above as [21]

$$\chi(\lambda) = 1 + \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \lambda_1}P + \frac{1/\bar{\lambda}_1 - 1/\lambda_1}{\lambda - 1/\bar{\lambda}_1}Q, \quad (3.11)$$

where P is the hermitian projection operator whose image is spanned by $\Psi(\bar{\lambda}_1)w_1$ for any constant four-component complex vector w_1 (the overall scale of w_1 clearly drops out so it parametrizes \mathbf{P}^3) and Q is the hermitian projection operator whose image is spanned by $\Psi(1/\lambda_1)\mathcal{J}\bar{w}_1$. Concretely,

$$P = \frac{\Psi(\bar{\lambda}_1)w_1w_1^\dagger [\Psi(\bar{\lambda}_1)]^\dagger}{w_1^\dagger [\Psi(\bar{\lambda}_1)]^\dagger \Psi(\bar{\lambda}_1)w_1}, \quad Q = \frac{\Psi(1/\lambda_1)\mathcal{J}\bar{w}_1w_1^\dagger \mathcal{J}^{-1} [\Psi(1/\lambda_1)]^\dagger}{w_1^\dagger \mathcal{J}^{-1} [\Psi(1/\lambda_1)]^\dagger \Psi(1/\lambda_1)\mathcal{J}\bar{w}_1}. \quad (3.12)$$

Then

$$\Psi'(\lambda) = \chi(\lambda)\Psi(\lambda) \quad (3.13)$$

satisfies the constraints (3.9) and (3.10), and provides the desired one-soliton solution $g' = \Psi'(0)$ to the original equations (3.1) and (3.2). Unlike the $\text{SO}(6)/\text{SO}(5)$ coset considered in [8], in this case there are no restrictions on the complex polarization vector w_1 . Repeated application of this procedure can be used to generate multi-soliton solutions.

4. Giant magnons on $\mathbb{R} \times S^5$

To apply the dressing method we begin with the vacuum solution

$$Z_1 = e^{it}, \quad Z_2 = 0, \quad Z_3 = 0, \quad (4.1)$$

which describes a point-like string moving at the speed of light around the equator of the S^5 . This state clearly has $\Delta - J = 0$. After embedding this solution into $\text{SU}(4)$ as in (3.6), a simple calculation reveals that the desired solution $\Psi(\lambda)$ to the linear system (3.7) subject to the constraints (3.8)–(3.10) is

$$\Psi(\lambda) = \text{diag}(e^{+iZ(\lambda)}, e^{-iZ(\lambda)}, e^{+iZ(\lambda)}, e^{-iZ(\lambda)}), \quad Z(\lambda) = \frac{z}{\lambda - 1} + \frac{\bar{z}}{\lambda + 1}. \quad (4.2)$$

4.1 A single dyonic giant magnon

Let us begin by applying the dressing method once to the vacuum (4.1). We will reproduce the dyonic giant magnon solution of [3]. The value of this exercise is to set some notation for subsequent solutions and also to illustrate the physical significance of the parameters λ_1 and w_1 which characterize the soliton. We parametrize the latter as

$$w_1 = \begin{pmatrix} +ie^{+y_1/2+i\psi_1/2+i\chi_1/2} \cos \alpha_1 \\ e^{-y_1/2-i\psi_1/2+i\chi_1/2} \cos \beta_1 \\ -ie^{+y_1/2-i\psi_1/2-i\chi_1/2} \sin \alpha_1 \\ e^{-y_1/2+i\psi_1/2-i\chi_1/2} \sin \beta_1 \end{pmatrix}, \quad (4.3)$$

where y_1 is complex and the remaining four angles are real. Here we have used the fact that the overall scale of w_1 drops out. Application of the dressing method gives the solution

$$\begin{aligned} Z_1 &= \frac{e^{+it}}{|\lambda_1|} \left[\frac{\bar{\lambda}_1 e^{-2iZ(\lambda_1)+\bar{y}_1}}{\mathcal{D}_1} + \frac{\lambda_1 e^{+2iZ(\lambda_1)-\bar{y}_1}}{\bar{\mathcal{D}}_1} \right], \\ Z_2 &= \frac{ie^{i\psi_1}(\bar{\lambda}_1 - \lambda_1)}{|\lambda_1|} \left[\frac{e^{-it} \cos \alpha_1 \cos \beta_1}{\mathcal{D}_1} + \frac{e^{+it} \sin \alpha_1 \sin \beta_1}{\bar{\mathcal{D}}_1} \right], \\ Z_3 &= \frac{ie^{i\chi_1}(\bar{\lambda}_1 - \lambda_1)}{|\lambda_1|} \left[\frac{e^{-it} \cos \alpha_1 \sin \beta_1}{\mathcal{D}_1} - \frac{e^{+it} \sin \alpha_1 \cos \beta_1}{\bar{\mathcal{D}}_1} \right], \end{aligned} \quad (4.4)$$

where

$$\mathcal{D}_1 = e^{-2iZ(\lambda_1)+\bar{y}_1} + e^{-2iZ(\bar{\lambda}_1)-y_1}. \quad (4.5)$$

The solution (4.4) carries U(1) charge

$$\Delta - J = \frac{\sqrt{\lambda}}{4\pi} \left| \lambda_1 - \bar{\lambda}_1 - \frac{1}{\lambda_1} + \frac{1}{\bar{\lambda}_1} \right| \quad (4.6)$$

and one non-zero SO(4) angular momentum

$$J_1 = \frac{\sqrt{\lambda}}{4\pi} \left| \lambda_1 - \bar{\lambda}_1 + \frac{1}{\lambda_1} - \frac{1}{\bar{\lambda}_1} \right|. \quad (4.7)$$

Note that $\Delta - J$ is always strictly positive, but we have defined J_1 to be positive by choice—the eigenvalues of (2.6) come in \pm pairs. Furthermore, the value of p for this solution, which may be read off by comparing (4.4) to (2.4), is given by

$$e^{ip} = \frac{\lambda_1}{\bar{\lambda}_1}. \quad (4.8)$$

In fact, λ_1 and $\bar{\lambda}_1$ are (sometimes up to an author-dependent normalization factor) the quantities frequently referred to in the recent literature as x^+ and x^- (see in particular [23, 24]). From the worldsheet point of view, the solution (4.4) describes a wave which propagates with phase velocity (i.e., the waveform depends on $x - v_1 t$) given by

$$v_1 = \frac{\lambda_1 + \bar{\lambda}_1}{1 + |\lambda_1|^2}. \quad (4.9)$$

Using (4.6), (4.7) and (4.8), we find that the dispersion relation takes the familiar form for the dyonic giant magnon [2, 3]

$$\Delta - J = \sqrt{J_1^2 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}}. \quad (4.10)$$

Note that all of the parameters associated with the choice of the ‘polarization’ w_1 completely drop out of the expressions for the conserved charges and the dispersion relation.

We can be more explicit about the role of these parameters. The parameters α_1 , β_1 , ψ_1 and χ_1 determine the ‘orientation’ of the soliton in the transverse \mathbb{R}^4 . Specifically, the angular momentum matrix (2.6) has eigenvalues $(+J_1, -J_1, 0, 0)$, so the soliton is characterized by an amount J_1 of angular momentum inside a certain 2-plane in the transverse \mathbb{R}^4 . The four parameters α_1 , β_1 , ψ_1 and χ_1 label the particular plane (they are coordinates on the Grassmannian $\text{Gr}_2(\mathbb{R}^4)$).

The remaining complex parameter y_1 can be completely absorbed by making the translation

$$Z(\lambda_1) \rightarrow Z(\lambda_1) - \frac{i}{2} \bar{y}_1. \quad (4.11)$$

The real part of y_1 corresponds to a translation of the soliton in the x direction while the imaginary part of y_1 corresponds to a rotation inside the plane of the soliton’s angular momentum.

4.2 A scattering state of two dyonic magnons, with three spins on S^5

Having analyzed in detail the parameter space for a single soliton in the last section, we are now in a position to use the dressing method to construct multi-soliton scattering states. The general n -soliton solution is specified by n complex numbers λ_i which encode the energy (4.6) and angular momentum (4.7) of each soliton. Each soliton with non-zero angular momentum is also characterized by the choice of a 2-plane inside the transverse \mathbf{R}^4 . Finally, the n -soliton solution has a non-obvious classical shift symmetry of the form (4.11) for each i . For n solitons this gives an additional $2n$ real moduli, but 2 linear combinations can be absorbed into overall t and x translations.

The procedure for constructing an n -soliton solution is therefore clear, but generic solutions are rather messy. We display here an explicit formula only for the special case of two dyonic giant magnons with completely orthogonal angular momenta, specifically, with soliton 1's angular momentum in the Z_2 plane and soliton 2's angular momentum in the Z_3 plane. To this end we pick the polarization vectors

$$w_1^T = (i \quad 1 \quad 0 \quad 0), \quad w_2^T = (i \quad 0 \quad 0 \quad 1). \quad (4.12)$$

Applying the dressing method twice with parameters (λ_1, w_1) and then (λ_2, w_2) gives the solution

$$\begin{aligned} Z_1 &= \frac{e^{+it} \mathcal{N}_{12}}{|\lambda_1 \lambda_2| \mathcal{D}_{12}}, \\ Z_2 &= \frac{ie^{-it}(\bar{\lambda}_1 - \lambda_1)\lambda_2}{|\lambda_1 \lambda_2|} \left[\frac{\lambda_1 \bar{\lambda}_2 - 1}{\lambda_1 \lambda_2 - 1} e^{2iZ(\lambda_2)} + \frac{\bar{\lambda}_1 - \bar{\lambda}_2}{\bar{\lambda}_1 - \lambda_2} e^{2iZ(\bar{\lambda}_2)} \right] \frac{e^{2i(Z(\lambda_1) + Z(\bar{\lambda}_1))}}{\mathcal{D}_{12}}, \\ Z_3 &= \frac{ie^{-it}(\bar{\lambda}_2 - \lambda_2)\lambda_1}{|\lambda_1 \lambda_2|} \left[\frac{\bar{\lambda}_1 \lambda_2 - 1}{\lambda_1 \lambda_2 - 1} e^{2iZ(\lambda_1)} + \frac{\bar{\lambda}_1 - \bar{\lambda}_2}{\lambda_1 - \bar{\lambda}_2} e^{2iZ(\bar{\lambda}_1)} \right] \frac{e^{2i(Z(\lambda_2) + Z(\bar{\lambda}_2))}}{\mathcal{D}_{12}}, \end{aligned} \quad (4.13)$$

where

$$\begin{aligned} \mathcal{N}_{12} &= (\lambda_1 e^{2iZ(\lambda_1)} \quad \bar{\lambda}_1 e^{2iZ(\bar{\lambda}_1)}) \begin{pmatrix} \left| \frac{\lambda_1 \bar{\lambda}_2 - 1}{\lambda_1 \lambda_2 - 1} \right|^2 & 1 \\ 1 & \left| \frac{\lambda_1 - \lambda_2}{\bar{\lambda}_1 - \lambda_2} \right|^2 \end{pmatrix} \begin{pmatrix} \lambda_2 e^{2iZ(\lambda_2)} \\ \bar{\lambda}_2 e^{2iZ(\bar{\lambda}_2)} \end{pmatrix}, \\ \mathcal{D}_{12} &= (e^{2iZ(\lambda_1)} \quad e^{2iZ(\bar{\lambda}_1)}) \begin{pmatrix} \left| \frac{\lambda_1 \bar{\lambda}_2 - 1}{\lambda_1 \lambda_2 - 1} \right|^2 & 1 \\ 1 & \left| \frac{\lambda_1 - \lambda_2}{\bar{\lambda}_1 - \lambda_2} \right|^2 \end{pmatrix} \begin{pmatrix} e^{2iZ(\lambda_2)} \\ e^{2iZ(\bar{\lambda}_2)} \end{pmatrix}, \end{aligned} \quad (4.14)$$

and $Z(\lambda)$ is defined in (4.2).

This solution carries U(1) charge

$$\Delta - J = \frac{\sqrt{\lambda}}{4\pi} \sum_{i=1}^2 \left| \lambda_i - \bar{\lambda}_i - \frac{1}{\lambda_i} + \frac{1}{\bar{\lambda}_i} \right| \quad (4.15)$$

and two independent angular momenta

$$J_i = \frac{\sqrt{\lambda}}{4\pi} \left| \lambda_i - \bar{\lambda}_i + \frac{1}{\lambda_i} - \frac{1}{\bar{\lambda}_i} \right|, \quad (4.16)$$

which are the eigenvalues of the angular momentum matrix (2.6) in the Z_2 and Z_3 planes respectively. The total momentum of this giant magnon is

$$e^{ip} = e^{i(p_1+p_2)} = \frac{\lambda_1 \lambda_2}{\bar{\lambda}_1 \bar{\lambda}_2}, \tag{4.17}$$

and the dispersion relation can be written as

$$\Delta - J = \sqrt{J_1^2 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_1}{2}} + \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_2}{2}}. \tag{4.18}$$

Special cases of the solution (4.13) have appeared previously in the literature. The case $p_1 = -p_2 = \pi$ was presented in [8], and a generalization to $p_1 = -p_2$ was given in [9]. Making direct contact with the former requires shifting $u_1 \rightarrow u_1 - c$ and $u_2 \rightarrow u_2 + c$ in (5.19) of [8], where

$$\tanh c = \left| \frac{\lambda_2}{\lambda_1} \right| \frac{|\lambda_1|^2 - 1}{|\lambda_2|^2 - 1}. \tag{4.19}$$

4.3 A scattering state of two HM magnons, with arbitrary positions on the transverse S^3

In the previous subsection we chose the particular polarization vectors (4.12) in order to avoid too much clutter in (4.13). An interesting limit in which the formulas simplify is when $|\lambda_i| \rightarrow 1$. Taking λ_i onto the unit circle sets the angular momentum of each soliton to zero — the dyonic giant magnon reduces to the elementary Hofman-Maldacena magnon [1]. Each such magnon is characterized by a momentum p and a unit vector n^a in the transverse \mathbf{R}^4 which specifies the polarization of its fluctuation away from the equator of the S^5 . Using the real basis defined under (2.6), the solution for such a scattering state can be written as

$$\begin{aligned} Z_1 &= e^{it} + \frac{e^{it}}{\mathcal{D}_{12}} \left[\cos \frac{p_1}{2} - \cos \frac{p_2}{2} + i \sin \frac{p_1}{2} \tanh u_1 - i \sin \frac{p_2}{2} \tanh u_2 \right], \\ X^a &= \frac{1}{\mathcal{D}_{12}} \left[n_1^a \sin \frac{p_1}{2} \operatorname{sech} u_1 - n_2^a \sin \frac{p_2}{2} \operatorname{sech} u_2 \right], \quad a = 1, 2, 3, 4, \end{aligned} \tag{4.20}$$

where

$$u_i = i(Z(\lambda_i) - Z(\bar{\lambda}_i)) = (x - t \cos \frac{p_i}{2}) \operatorname{csc} \frac{p_i}{2} \tag{4.21}$$

and now

$$\mathcal{D}_{12} = \frac{1 - \cos \frac{p_1}{2} \cos \frac{p_2}{2} - \sin \frac{p_1}{2} \sin \frac{p_2}{2} [\tanh u_1 \tanh u_2 + (n_1 \cdot n_2) \operatorname{sech} u_1 \operatorname{sech} u_2]}{\cos \frac{p_2}{2} - \cos \frac{p_1}{2}}. \tag{4.22}$$

It is also straightforward to obtain this solution via the Bäcklund transformation (see [25] in particular). The conserved charges and dispersion relation of this solution do not depend on the polarization vectors n_i^a .

5. Classical time delay for scattering of dyonic magnons

With explicit formulas for the scattering solutions in hand, it is a simple matter to read off the classical time delay for soliton scattering. To find the time delay as soliton 1 passes soliton 2 (let us take $v_1 > v_2 > 0$ with the velocities v_i given by (4.9)) we set $x = v_1(t - \delta t)$ and compare the solution at $t \rightarrow \pm\infty$ to the single soliton solution. The total time delay is then $\Delta T_{12} = \delta t_+ - \delta t_-$.

We are particularly interested in seeing the dependence of the time delay on the relative orientations of the angular momenta of the two scattering solitons. Without loss of generality we can take the polarization of w_1 as in (4.12), but we keep w_2 arbitrary as in (4.3). We find

$$\Delta T_{12} = \frac{i}{2} \frac{|1 - \lambda_1|^2 |1 + \lambda_1|^2}{\lambda_1^2 - \bar{\lambda}_1^2} \log [(A \cos^2 \alpha_2 + B \sin^2 \alpha_2)(A \cos^2 \beta_2 + B \sin^2 \beta_2)], \quad (5.1)$$

where

$$A = \frac{|\lambda_1 - \lambda_2|^2}{|\lambda_1 - \bar{\lambda}_2|^2}, \quad B = \frac{|\lambda_1 - 1/\bar{\lambda}_2|^2}{|\lambda_1 - 1/\lambda_2|^2}. \quad (5.2)$$

It would be interesting to evaluate the corresponding classical phase shift δ_{12} (i.e., the S -matrix element $e^{i\delta_{12}}$), which may be obtained by integrating ΔT_{12} with respect to the energy of soliton 1 (4.6) while holding the angular momentum (4.7) fixed, and to compare the result with a corresponding gauge theory calculation along the lines of [13, 14].

We can subject (5.1) to some consistency checks by comparing special cases of the formula to known results. First of all, we can recover the scattering of two HM magnons by taking $\lambda_i = e^{ip_i/2}$ on the unit circle. In this case we obtain

$$\Delta T_{12} = \tan \frac{p_1}{2} \log \left[\frac{1 - \cos \frac{1}{2}(p_1 - p_2)}{1 - \cos \frac{1}{2}(p_1 + p_2)} \right], \quad (5.3)$$

in complete agreement with the result of [1]. Note that this result is independent of the positions of the two magnons on the transverse S^3 , in accord with the expectation of [1]. The result (5.3) can also be read off directly from the solution (4.20).

Another check is obtained by setting $\alpha_2 = \beta_2 = 0$ so that we have two dyonic giant magnons whose angular momenta both lie within the Z_2 plane, leading to

$$\Delta T_{12} = 2i \frac{|1 - \lambda_1|^2 |1 + \lambda_1|^2}{\lambda_1^2 - \bar{\lambda}_1^2} \log \left| \frac{\lambda_1 - \lambda_2}{\lambda_1 - \bar{\lambda}_2} \right| \quad (5.4)$$

in complete agreement with [13, 14] where this case was recently studied.

As a final consistency check, we can take $\alpha_2 = \beta_2 = \pi/2$, which leads to

$$\Delta T_{12} = 2i \frac{|1 - \lambda_1|^2 |1 + \lambda_1|^2}{\lambda_1^2 - \bar{\lambda}_1^2} \log \left| \frac{\lambda_1 - 1/\bar{\lambda}_2}{\lambda_1 - 1/\lambda_2} \right|. \quad (5.5)$$

From equation (4.4) it is evident that this choice of orientation simply reverses the sign of the angular momentum of the second soliton relative to $\alpha = \beta = 0$. But this is completely equivalent to changing $\lambda_2 \rightarrow 1/\bar{\lambda}_2$, which is indeed precisely the transformation between (5.4) and (5.5).

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